数理解析研究所講究録 1414 巻 2005 年 46-58

Some Coefficient Inequalities and Distortion Bounds Associated with Certain New Subclasses of Analytic Functions

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Abstract

The authors introduce and investigate two new subclasses $\mathcal{M}^*(\alpha)$ and $\mathcal{N}^*(\alpha)$ of normalized analytic functions satisfying certain coefficient inequalities in the open unit disk $\mathbb{U}$. The main results of the present paper provide various interesting properties of functions belonging to the classes $\mathcal{M}^*(\alpha)$ and $\mathcal{N}^*(\alpha)$. Some of these properties include (for example) several coefficient inequalities, distortion bounds and inclusion relationships for the function classes which are considered here.

2000 *Mathematics Subject Classification.* Primary 30C45; Secondary 30A10, 30C50.
Key Words and Phrases. Coefficient inequalities, distortion bounds, analytic functions, univalent functions, starlike functions, convex functions, integro-differential operator, inclusion relationships, monotonically increasing and monotonically decreasing functions.

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ *normalized* in the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}. \tag{1.2}$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which are also *univalent* in $\mathbb{U}$.

Let $\mathcal{S}^*(\alpha)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U};\ 0 \leqq \alpha < 1). \tag{1.3}$$

A function $f \in \mathcal{S}^*(\alpha)$ is said to be *starlike of order* α in $\mathbb{U}$. Furthermore, let $\mathcal{K}(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy the following inequality:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U};\ 0 \leqq \alpha < 1). \tag{1.4}$$

A function $f \in \mathcal{K}(\alpha)$ is said to be *convex of order* α in $\mathbb{U}$. We note that

$$f(z) \in \mathcal{K}(\alpha) \quad \Longleftrightarrow \quad zf'(z) \in \mathcal{S}^*(\alpha).$$

(See, for details, [1] and [2]; see also [3] and [6], and the references cited therein.)

About three decades ago, Silverman [5] gave the following coefficient inequalities for the function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$.

Theorem A (Silverman [5]). *If $f(z) \in \mathcal{A}$ satisfies the following coefficient inequality:*

$$\sum_{n=2}^{\infty}(n - \alpha)|a_n| \leqq 1 - \alpha \quad (0 \leqq \alpha < 1), \tag{1.5}$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \alpha \quad (z \in \mathbb{U};\ 0 \leqq \alpha < 1), \tag{1.6}$$

that is, that $f(z) \in \mathcal{S}^*(\alpha)$.

Theorem B (Silverman [5]). *If $f(z) \in \mathcal{A}$ satisfies the following coefficient inequality:*

$$\sum_{n=2}^{\infty}n(n - \alpha)|a_n| \leqq 1 - \alpha \quad (0 \leqq \alpha < 1), \tag{1.7}$$

then

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1 - \alpha \quad (z \in \mathbb{U};\ 0 \leqq \alpha < 1), \tag{1.8}$$

that is, that $f(z) \in \mathcal{K}(\alpha)$.

More recently, Sekine and Owa [4] considered the subclass of functions $f \in \mathcal{A}$ which satisfy the following inequality:

$$\left|\frac{zf'(z)}{f(z)} - a\right| < a - \alpha \quad (z \in \mathbb{U};\ 0 \leqq \alpha < 1;\ a > \alpha). \tag{1.9}$$

In this paper, we consider a new subclass $\mathcal{M}(\alpha)$ of the class $\mathcal{A}$ consisting of functions $f(z)$ such that

$$\left|\frac{f(z)}{zf'(z)} - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha} \quad (z \in \mathbb{U};\ 0 < \alpha < 1). \tag{1.10}$$

We also introduce and investigate here the subclass $\mathcal{N}(\alpha)$ of the class $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the following inclusion relationship:

$$zf'(z) \in \mathcal{M}(\alpha).$$

Let us now define the function $F(z)$ by

$$F(z) = \frac{zf'(z)}{f(z)} \quad (f \in \mathcal{M}(\alpha)).$$

Then $f(z)$ satisfies the inequality:

$$F(z) + \bar{F}(z) > 2\alpha \quad (z \in \mathbb{U};\ 0 < \alpha < 1), \tag{1.11}$$

so that

$$\Re(F(z)) = \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U};\ 0 < \alpha < 1). \tag{1.12}$$

It follows from (1.12) that

$$\mathcal{M}(\alpha) \subset \mathcal{S}^*(\alpha) \quad \text{and} \quad \mathcal{N}(\alpha) \subset \mathcal{K}(\alpha).$$

Example. Let us consider the function given by

$$f(z) = z + \frac{1}{k}z^2 \quad (k \geqq 2). \tag{1.13}$$

Then we have

$$\frac{zf'(z)}{f(z)} - 1 = \frac{k+2z}{k+z} - 1 = \frac{z}{k+z}. \tag{1.14}$$

Since

$$\left|\frac{z}{k+z} + \frac{1}{k^2-1}\right| < \frac{k}{k^2-1} \quad (z \in \mathbb{U}), \tag{1.15}$$

we see that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{1}{k-1} = 1 - \frac{k-2}{k-1} \tag{1.16}$$

which readily implies that

$$f(z) \in \mathcal{S}^*\left(\frac{k-2}{k-1}\right). \tag{1.17}$$

On the other hand, we observe that

$$\frac{f(z)}{zf'(z)} - \frac{1}{2\alpha} = \frac{k+z}{k+2z} - \frac{1}{2\alpha} = \frac{1}{2}\left(1 - \frac{1}{\alpha} + \frac{k}{k+2z}\right). \tag{1.18}$$

Noting also that

$$\left|\frac{k}{k+2z} - \frac{k^2}{k^2-4}\right| < \frac{2k}{k^2-4} \quad (z \in \mathbb{U}), \tag{1.19}$$

we have

$$\left|\frac{f(z)}{zf'(z)} - \frac{1}{2\alpha}\right| < \frac{1}{2}A(k,\alpha)\,, \tag{1.20}$$

where

$$A(k,\alpha) := \max\left\{\left|1 - \frac{1}{\alpha} + \frac{k}{k+2}\right|, \left|1 - \frac{1}{\alpha} + \frac{k}{k-2}\right|\right\}. \tag{1.21}$$

Thus we obtain

$$\frac{1}{\alpha} = \frac{1}{2}A(k,\alpha) \tag{1.22}$$

for $f(z) \in \mathcal{M}(\alpha)$. Let us put $\alpha = \alpha_0$. If α is given by

$$\alpha = \frac{1}{A(k,\alpha)}, \tag{1.23}$$

then $f(z) \in \mathcal{M}(\alpha_0)$. By the fact that $\mathcal{M}(\alpha) \subset \mathcal{S}^*(\alpha)$, we have

$$\alpha_0 \geqq \frac{k-2}{k-1}. \tag{1.24}$$

If we set

$$\alpha = \frac{k-2}{k-1}, \tag{1.25}$$

then we have

$$1 - \frac{1}{\alpha} + \frac{k}{k+2} = 1 - \frac{k-1}{k-2} + \frac{k}{k+2} = \frac{(k+1)(k-4)}{(k+2)(k-2)} \tag{1.26}$$

and

$$1 - \frac{1}{\alpha} + \frac{k}{k-2} = 1 - \frac{k-1}{k-2} + \frac{k}{k-2} = \frac{k-1}{k-2}. \tag{1.27}$$

Therefore, in the case when $k \geqq 4$, we have

$$\frac{k-1}{k-2} - \frac{(k+1)(k-4)}{(k+2)(k-2)} = \frac{2(2k+1)}{(k+2)(k-2)} \geqq 0. \tag{1.28}$$

Moreover, in the case when $2 \leqq k < 4$, we have

$$\frac{k-1}{k-2} - \frac{(k+1)(4-k)}{(k+2)(k-2)} = \frac{2(k^2-k-3)}{(k+2)(k-2)}. \tag{1.29}$$

Thus, if

$$2 \leqq k \leqq \frac{1+\sqrt{13}}{2} = 2.3027\cdots,$$

then we have

$$\frac{k-1}{k-2} \leqq \frac{(k+1)(4-k)}{(k+2)(k-2)}. \tag{1.30}$$

Therefore

$$A(k,\alpha) = \begin{cases} \dfrac{k-1}{k-2} & \left(k \geqq \dfrac{1+\sqrt{13}}{2}\right) \\ \dfrac{(k+1)(4-k)}{(k+2)(k-2)} & \left(2 \leqq k < \dfrac{1+\sqrt{13}}{2}\right). \end{cases} \tag{1.31}$$

By the condition (1.23), we have

$$\alpha_0 = \frac{k-2}{k-1} \qquad \left(k \geqq \frac{1+\sqrt{13}}{2}\right) \tag{1.32}$$

such that

$$f(z) \in \mathcal{M}\left(\frac{k-2}{k-1}\right) \qquad \left(k \geqq \frac{1+\sqrt{13}}{2} = 2.3027\cdots\right). \tag{1.33}$$

Thus we have

$$\frac{5-\sqrt{13}}{6} \leqq \frac{k-2}{k-1} < 1 \qquad \left(\frac{5-\sqrt{13}}{6} = 0.23241\cdots\right). \tag{1.34}$$

When $0 < \alpha \leqq \beta < 1$, we have the following inclusion relationship:

$$\mathcal{M}(\alpha) \supset \mathcal{M}(\beta), \tag{1.35}$$

which results from the definition of the class $\mathcal{M}(\alpha)$. Thus we conclude that

$$f(z) \in \mathcal{M}\left(\frac{k-2}{k-1}\right) \subset \mathcal{M}\left(\frac{5-\sqrt{13}}{6}\right) \subset \mathcal{S}^*\left(\frac{5-\sqrt{13}}{6}\right). \tag{1.36}$$

We now consider the following function:

$$f(z) = z + \frac{1}{2k}z^2 \quad (k \geqq 2), \tag{1.37}$$

which immediately yields

$$zf'(z) = z + \frac{1}{k}z^2 \quad (k \geqq 2). \tag{1.38}$$

Since, by definition,

$$f(z) \in \mathcal{N}(\alpha) \iff zf'(z) \in \mathcal{M}(\alpha),$$

we finally obtain the following inclusion relationship:

$$f(z) \in \mathcal{N}\left(\frac{k-2}{k-1}\right) \subset \mathcal{N}\left(\frac{5-\sqrt{13}}{6}\right) \subset \mathcal{K}\left(\frac{5-\sqrt{13}}{6}\right) \tag{1.39}$$

$$\left(k \geqq \frac{1+\sqrt{13}}{2} = 2.3027\cdots\right).$$

2 A Set of Coefficient Inequalities

Our first coefficient inequality is contained in Theorem 1 below.

Theorem 1. *Let* $0 < \alpha < 1$. *If* $f(z) \in \mathcal{A}$ *satisfies the following coefficient inequality:*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leqq \frac{1}{2}(1-|1-2\alpha|) = \begin{cases} \alpha & \left(0 < \alpha \leqq \frac{1}{2}\right) \\ 1-\alpha & \left(\frac{1}{2} \leqq \alpha < 1\right), \end{cases} \tag{2.1}$$

then $f(z) \in \mathcal{M}(\alpha)$.

Proof. By virtue of the condition (1.10), we have to show that

$$\left|\frac{2\alpha f(z)}{zf'(z)} - 1\right| < 1. \tag{2.2}$$

We first observe that

$$\left|\frac{2\alpha f(z) - zf'(z)}{zf'(z)}\right| = \left|\frac{1 - 2\alpha + \sum_{n=2}^{\infty}(n-2\alpha)a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}}\right| \tag{2.3}$$

$$\leqq \frac{|1-2\alpha| + \sum_{n=2}^{\infty}(n-2\alpha)|a_n| \cdot |z|^{n-1}}{1 - \sum_{n=2}^{\infty} n|a_n| \cdot |z|^{n-1}}$$

$$< \frac{|1-2\alpha| + \sum_{n=2}^{\infty}(n-2\alpha)|a_n|}{1 - \sum_{n=2}^{\infty} n|a_n|}.$$

Now, by using the coefficient inequality (2.1), we have

$$\frac{|1-2\alpha| + \sum_{n=2}^{\infty}(n-2\alpha)|a_n|}{1 - \sum_{n=2}^{\infty} n|a_n|} \leqq 1, \tag{2.4}$$

which, in conjunction with (2.2) and (2.3), completes the proof of Theorem 1. □

By means of Theorem 1, we introduce the subclass $\mathcal{M}^*(\alpha)$ of the class $\mathcal{M}(\alpha)$ consisting of all functions $f(z)$ which satisfy the coefficient inequality (2.1) for some α $(0 < \alpha < 1)$.

Theorem 2. *Suppose that* $0 < \alpha < 1$. *If* $f(z) \in \mathcal{A}$ *satisfies the following coefficient inequality:*

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leqq \frac{1}{2}(1 - |1-2\alpha|) = \begin{cases} \alpha & \left(0 < \alpha \leqq \frac{1}{2}\right) \\ 1-\alpha & \left(\frac{1}{2} \leqq \alpha < 1\right), \end{cases} \tag{2.5}$$

then $f(z) \in \mathcal{N}(\alpha)$.

Proof. The proof of Theorem 2 follows from Theorem 1 and the aforementioned fact that

$$f(z) \in \mathcal{N}(\alpha) \iff zf'(z) \in \mathcal{M}(\alpha).$$

□

By means of Theorem 2, we also introduce the subclass $\mathcal{N}^*(\alpha)$ of the class $\mathcal{N}(\alpha)$ consisting of all functions $f(z)$ which satisfy the coefficient inequality (2.5) for some α $(0 < \alpha < 1)$.

3 Distortion Bounds

For $f \in \mathcal{A}$, we define the integro-differential operators $I_k f(z)$ given by

$$I_{-1}f(z) = f'(z), \quad I_0 f(z) = f(z),$$

and

$$I_k f(z) = \int_0^z I_{k-1}f(t)dt \qquad (k \in \mathbb{N} := \{1, 2, 3, \cdots).$$

Then we find from (1.1) that

$$I_k f(z) = \frac{1}{(k+1)!} z^{k+1} + \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n z^{n+k}. \tag{3.1}$$

Theorem 3. *If $f(z) \in \mathcal{M}^*(\alpha)$, then*

$$\begin{aligned} \frac{1}{(k+1)!}|z|^{k+1} - \frac{1-|1-2\alpha|}{(k+2)!(2-\alpha)}|z|^{k+2} &\leqq |I_k f(z)| \\ &\leqq \frac{1}{(k+1)!}|z|^{k+1} + \frac{1-|1-2\alpha|}{(k+2)!(2-\alpha)}|z|^{k+2} \end{aligned} \tag{3.2}$$

$$(z \in \mathbb{U};\ k \in \mathbb{N}_0 \cup \{-1\};\ \mathbb{N}_0 := \mathbb{N} \cup \{0\}).$$

Proof. We begin by noting that

$$\begin{aligned} |I_k f(z)| &= \left| \frac{1}{(k+1)!} z^{k+1} + \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n z^{n+k} \right| \\ &\leqq \frac{1}{(k+1)!}|z|^{k+1} + \sum_{n=2}^{\infty} \frac{n!}{(n+k)!}|a_n| \cdot |z|^{n+k} \\ &< \frac{1}{(k+1)!}|z|^{k+1} + |z|^{k+2} \sum_{n=2}^{\infty} \frac{n!}{(n+k)!}|a_n|. \end{aligned} \tag{3.3}$$

Now it is easy to see that

$$\frac{(k+2)!(2-\alpha)}{2} \sum_{n=2}^{\infty} \frac{n!}{(n+k)!}|a_n| \leqq \sum_{n=2}^{\infty} (n - \alpha)|a_n| \leqq \frac{1}{2}(1 - |1 - 2\alpha|), \tag{3.4}$$

which implies that

$$\sum_{n=2}^{\infty} \frac{n!}{(n+k)!}|a_n| \leqq \frac{1-|1-2\alpha|}{(k+2)!(2-\alpha)}. \tag{3.5}$$

Therefore, we have

$$|I_k f(z)| \leqq \frac{1}{(k+1)!}|z|^{k+1} + \frac{1-|1-2\alpha|}{(k+2)!(2-\alpha)}|z|^{k+2} \quad (z \in \mathbb{U}). \tag{3.6}$$

Also we can easily observe that

$$|I_k f(z)| \geqq \frac{1}{(k+1)!}|z|^{k+1} - \frac{1-|1-2\alpha|}{(k+2)!(2-\alpha)}|z|^{k+2} \quad (z \in \mathbb{U}). \tag{3.7}$$

□

By setting $k = -1, 0, 1$ in Theorem 3, we deduce Corollary 1 below.

Corollary 1. *If $f(z) \in \mathcal{M}^*(\alpha)$, then*

$$(3.8) \qquad 1 - \frac{1-|1-2\alpha|}{2-\alpha}|z| \leqq |f'(z)| \leqq 1 + \frac{1-|1-2\alpha|}{2-\alpha}|z| \qquad (k=-1),$$

$$(3.9) \qquad |z| - \frac{1-|1-2\alpha|}{2(2-\alpha)}|z|^2 \leqq |f(z)| \leqq |z| + \frac{1-|1-2\alpha|}{2(2-\alpha)}|z|^2 \qquad (k=0),$$

and

$$(3.10) \qquad \frac{1}{2}|z|^2 - \frac{1-|1-2\alpha|}{6(2-\alpha)}|z|^3 \leqq |I_1 f(z)| \leqq \frac{1}{2}|z|^2 + \frac{1-|1-2\alpha|}{6(2-\alpha)}|z|^3 \qquad (k=1).$$

For $f \in \mathcal{A}$, we consider again the following integro-differential operators:

$$I_{-2}f(z) = f''(z) = \sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}, \quad I_{-1}f(z) = f'(z), \quad I_0 f(z) = f(z),$$

and

$$I_k f(z) = \int_0^z I_{k-1} f(t)dt \qquad (k \in \mathbb{N}).$$

Next we state and prove the following result.

Theorem 4. *If $f(z) \in \mathcal{N}^*(\alpha)$, then*

$$(3.11) \qquad 2|a_2| - \frac{1-|1-2\alpha|}{2}|z| \leqq |I_{-2}f(z)| \leqq 2|a_2| + \frac{1-|1-2\alpha|}{2}|z|$$

and

$$(3.12) \qquad \frac{1}{(k+1)!}|z|^{k+1} - \frac{1-|1-2\alpha|}{(k+2)!(2-\alpha)}|z|^{k+2} \leqq |I_k f(z)|$$

$$\leqq \frac{1}{(k+1)!}|z|^{k+1} + \frac{1-|1-2\alpha|}{(k+2)!(2-\alpha)}|z|^{k+2} \qquad (z \in \mathbb{U};\ k \in \mathbb{N}_0 \cup \{-1\}).$$

Proof. We note that, for $k \in \mathbb{N}_0 \cup \{-1\}$,

$$(3.13) \qquad |I_k f(z)| = \left| \frac{1}{(k+1)!} z^{k+1} + \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n z^{n+k} \right|$$

$$\leqq \frac{1}{(k+1)!}|z|^{k+1} + \sum_{n=2}^{\infty} \frac{n!}{(n+k)!}|a_n| \cdot |z|^{n+k}$$

$$< \frac{1}{(k+1)!}|z|^{k+1} + |z|^{k+2} \sum_{n=2}^{\infty} \frac{n!}{(n+k)!}|a_n|.$$

Since, for $f(z) \in \mathcal{N}^*(\alpha)$,

$$(3.14) \qquad (k+2)!(2-\alpha)\sum_{n=2}^{\infty}\frac{n!}{(n+k)!}|a_n| \leqq \sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leqq \frac{1}{2}(1-|1-2\alpha|),$$

we find that

$$(3.15) \qquad \sum_{n=2}^{\infty}\frac{n!}{(n+k)!}|a_n| \leqq \frac{1-|1-2\alpha|}{2(k+2)!(2-\alpha)}.$$

Therefore, we have

$$(3.16) \qquad \frac{1}{(k+1)!}|z|^{k+1} - \frac{1-|1-2\alpha|}{2(k+2)!(2-\alpha)}|z|^{k+2} \leqq |I_k f(z)|$$

$$\leqq \frac{1}{(k+1)!}|z|^{k+1} + \frac{1-|1-2\alpha|}{2(k+2)!(2-\alpha)}|z|^{k+2}$$

$$(z \in \mathbb{U};\ k \in \mathbb{N}_0 \cup \{-1\}).$$

In the exceptional case of (3.16) when $k = -2$, we have

$$(3.17) \qquad 2|a_2| - \frac{1-|1-2\alpha|}{2}|z| \leqq |I_{-2} f(z)| \leqq 2|a_2| + \frac{1-|1-2\alpha|}{2}|z|$$

$$(z \in \mathbb{U}).$$

□

By setting $k = -1, 0, 1$ in Theoem 4, we deduce the following corollary.

Corollary 2. *If $f(z) \in \mathcal{N}^*(\alpha)$, then*

$$(3.18) \qquad 1 - \frac{1-|1-2\alpha|}{2(2-\alpha)}|z| \leqq |f'(z)| \leqq 1 + \frac{1-|1-2\alpha|}{2(2-\alpha)}|z| \qquad (k=-1),$$

$$(3.19) \qquad |z| - \frac{1-|1-2\alpha|}{4(2-\alpha)}|z|^2 \leqq |f(z)| \leqq |z| + \frac{1-|1-2\alpha|}{4(2-\alpha)}|z|^2 \qquad (k=0),$$

and

$$(3.20) \qquad \frac{1}{2}|z|^2 - \frac{1-|1-2\alpha|}{12(2-\alpha)}|z|^3 \leqq |I_1 f(z)| \leqq \frac{1}{2}|z|^2 + \frac{1-|1-2\alpha|}{12(2-\alpha)}|z|^3 \qquad (k=1).$$

4 Inclusion Relationships Between the Function Classes $\mathcal{M}^*(\alpha)$ and $\mathcal{N}^*(\alpha)$

Using the coefficient inequalities for the classes $\mathcal{M}^*(\alpha)$ and $\mathcal{N}^*(\alpha)$, we now derive Theorem 5 below.

Theorem 5. *The following inclusion relationships hold true for the class* $\mathcal{M}^*(\alpha)$:

(A) $\mathcal{M}^*(\alpha) \subset \mathcal{M}^*(1-\alpha) \qquad \left(0 < \alpha \leqq \frac{1}{2}\right).$

(B) $\mathcal{M}^*(\alpha) \subset \mathcal{M}^*\left(1-\frac{1}{3-2\alpha}\right) \qquad \left(\frac{1}{2} \leqq \alpha < 1\right).$

(C) $\mathcal{M}^*(\alpha) \subset \mathcal{M}^*(\beta) \qquad \left(0 < \alpha \leqq \beta \leqq \frac{1}{2}\right).$

(D) $\mathcal{M}^*(\beta) \subset \mathcal{M}^*(\alpha) \qquad \left(\frac{1}{2} \leqq \alpha \leqq \beta < 1\right).$

Proof. (A) For

$$0 < \alpha \leqq \frac{1}{2} \quad \text{and} \quad \frac{1}{2} \leqq \beta < 1,$$

we consider the maximum value of β such that

$$\sum_{n=2}^{\infty} \frac{n-\beta}{1-\beta}|a_n| \leqq \sum_{n=2}^{\infty} \frac{n-\alpha}{\alpha}|a_n| \leqq 1. \tag{4.1}$$

Thus we need to find the maximum value of β such that

$$\beta \leqq \frac{n(1-\alpha)-\alpha}{n-2\alpha} \qquad (n \in \mathbb{N}\setminus\{1\}). \tag{4.2}$$

By taking the derivative of the right-hand side of (4.2) with respect to n, it is easily seen that the right-hand side of (4.2) is monotonically decreasing for n. Thus, upon letting $n \to \infty$, we have $\beta = 1-\alpha$. Noting also that

$$\frac{1}{2} \leqq \beta < 1 \quad \text{for} \quad 0 < \alpha \leqq \frac{1}{2},$$

we have

$$\mathcal{M}^*(\alpha) \subset \mathcal{M}^*(1-\alpha) \qquad \left(0 < \alpha \leqq \frac{1}{2}\right),$$

which evidently completes the proof of (A).

The proofs of (B), (C), and (D) are much akin to the proof of (A).

□

Finally, we consider some relationships between the function classes $\mathcal{M}^*(\alpha)$ and $\mathcal{N}^*(\alpha)$.

Theorem 6. *Each of the following assertions holds true:*

(A) *If* $f(z) \in \mathcal{N}^*(\alpha)$ *for* $0 < \alpha \leqq \frac{1}{2}$, *then* $f(z) \in \mathcal{M}^*\left(\frac{4-4\alpha}{4-3\alpha}\right)$.

(B) *If* $f(z) \in \mathcal{N}^*(\alpha)$ *for* $\frac{1}{2} \leqq \alpha < 1$, *then* $f(z) \in \mathcal{M}^*\left(\frac{2-2\alpha}{5-3\alpha}\right)$.

(C) *If* $f(z) \in \mathcal{N}^*(\alpha)$ *for* $0 < \alpha \leqq \frac{1}{2}$, *then* $f(z) \in \mathcal{M}^*\left(\frac{2\alpha}{4-\alpha}\right)$.

(D) *If* $f(z) \in \mathcal{N}^*(\alpha)$ *for* $\frac{1}{2} \leqq \alpha < 1$, *then* $f(z) \in \mathcal{M}^*\left(\frac{2}{3-\alpha}\right)$.

Proof. (A) Let

$$0 < \alpha \leqq \frac{1}{2} \quad \text{and} \quad \frac{1}{2} \leqq \beta < 1.$$

We consider the maximum value of β such that

$$\sum_{n=2}^{\infty} \frac{n-\beta}{1-\beta} |a_n| \leqq \sum_{n=2}^{\infty} \frac{n(n-\alpha)}{\alpha} |a_n| \leqq 1. \tag{4.3}$$

This means that

$$\beta \leqq \frac{n^2 - 2n\alpha}{n^2 - n\alpha - \alpha} \quad (n \in \mathbb{N} \setminus \{1\}). \tag{4.4}$$

If we take the derivative of the right-hand side of (4.4) with respect to n, then the numerator becomes

$$n^2\alpha - 2n\alpha + 2\alpha^2 \geqq 0 \qquad \left(0 < \alpha \leqq \frac{1}{2};\, n \in \mathbb{N} \setminus \{1\}\right). \tag{4.5}$$

Therefore, the right-hand side of (4.4) is monotonically increasing for n.
Thus, by setting $n = 2$, we have

$$\beta = \frac{4-4\alpha}{4-3\alpha}.$$

It is easy to see that

$$\frac{1}{2} \leqq \beta < 1 \quad \text{for} \quad 0 < \alpha \leqq \frac{1}{2},$$

which obviously completes the proof of (A).

The proofs of (B), (C), and (D) would run parallel to the proof of (A).

□

Acknowledgements

The present investigation was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

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